THE KILLING-HOPF THEOREM

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Abstract. In this lecture , we discuss the celebrated Killing-Hopf theorem which classifies all complete and connected locally Euclidean Surfaces. Interestingly, it turns out that there are only four complete and connected locally Euclidean surfaces Viz. Cylinder, Torus, Twisted Cylinder and Kline Bottle ! The construction in the proof invloves the notion of quotient surfaces obatined by the action of certain subgrooups of isometry group of the Euclidean plane.

1. Introduction

The Euclidean plane is a surface of constant curvature and the curvature is zero everywhere. One would like to have a surface which locally looks like the euclidean plane. More precisely, one wants to find surfaces endowed with a metric which is locally euclidean in the sense that at every point has a neighbourhood which is locally isometric with a Euclidean neighbourhood of a point in $\mathbb{R}^2$. One may think of the surface $(Z = 0) \cup (Z = 1)$ or the punctured plane $\mathbb{R}^2 \backslash \{0\}$. Though these surfaces are not same as $\mathbb{R}^2$, locally they are euclidean when equipped with euclidean metric. Of course these examples are, in some sense, very trivial. In the sequel, more non-trivial examples have been constructed by using the action of certain subgroups of the group of isometries of the plane on the plane.

In section II, we briefly discuss the group of isometries of $\mathbb{R}^2$. In section III, we shall define the notion of locally euclidean surface and also give the constructiion of quotient surfaces which are loacally euclidean. Section IV deals with covering surfaces by the plane, covering isometry group and the proof of Killing-Hopf theorem.

2. The Group of Isometries of $\mathbb{R}^2$

The euclidean plane is the set $\mathbb{R}^2$ equipped with the inner product $\langle X, Y \rangle = X.Y$ where $X.Y$ denotes the dot product given by $x_1y_1 + x_2y_2$ where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. This innere product gives the standard eucliden metric on $\mathbb{R}^2$. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called an isometry if f preserves the euclidean distance, that is,

$$d(f(P), f(Q)) = d(P, Q), \ \forall P, Q \in \mathbb{R}^2$$

The following are typical examples of isometry.

Key words and phrases. Locally Euclidean surface, quotient surfaces, isometry group .

Example 2.1 : Translation by a verctor (α, β)

$$t_{(\alpha,\beta)} : (x, y) \to (x + \alpha, y + \beta)$$

Example 2.2 : Rotation about the origin O by an angle θ

$$r_\theta : (x, y) \to (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$

Example 2.3 : Reflection in a line L passing through origin and making an inclination θ with positive $X-$ axis

$$s_\theta : (x, y) \to (x \cos 2\theta + y \sin 2\theta, x \sin 2\theta - y \cos 2\theta)$$

We denote the set of isometries of $\mathbb{R}^2$ by $Iso(\mathbb{R}^2)$. Observe that $Iso(\mathbb{R}^2)$ is closed under the operation of composition of functions. Further, the idntity map is the identity element for this operation on $Iso(\mathbb{R}^2)$. Note that in each of the examples above, the isometry has inverse. Indeed,

$$t^{-1}_{(\alpha,\beta)} = t_{(-\alpha,-\beta)}, \quad r^{-1}_\theta = r_{-\theta}, \quad s^{-1}_\theta = s_\theta$$

Using the operation of composition, we can describe the rotation about an arbitrary point (a, b) through angle θ as the product of isometries $t^{-1}_{(a,b)} r_\theta t_{(a,b)}$. Such an element is called the conjugation of r_θ by $t_{(a,b)}$.

Exercise 2.4 : Express the reflection in an arbitrary line as the conjugation of s_θ.

Example 2.5 : Glide Reflection
The product of a reflection with a translation in the direction of the line of reflection is called a glide reflection with axis L. For example, $t_{(a,0)} r_0$ is the glide reflection with $X-$axis as the axis of reflection.

The examples of isometries described above are fundamental in nature. Especially, reflections turn out to be building blocks for all the isometries. We shall state some important peoperties of these isometries that allow us to express every isometry as a product of reflections.

Theorem 2.6.
(1) *Any translation or rotation is the product of two reflections*
(2) *The product of two reflections is a translation or rotation*
(3) *The set of translations and rotations is closed under the product.*

Remark 2.7 : A more concise way of computing with isometries is to express them as complex functions of one variable. We can consider a point $(x, y) \in \mathbb{R}^2$ as $z = x + iy \in \mathbb{C}$. Then, $t_{(\alpha,\beta)}$ becomes the function $t_\alpha + i\beta(z) = \alpha + i\beta + z$, r_θ becomes $r_\theta(z) = e^{i\theta} z$ and s_0 becomes $s_0(z) = \overline{z}$. This simplifies products of isometries.

Exercise 2.8 : Show that any line $ax + by + c = 0$ is the set of points equidistant from two suitably chosen points (x_1, x_2) and (y_1, y_2). Hence, conclude (without assuming all isometries are generated by $t_{(\alpha,\beta)}, r_\theta$ and s_θ) that all isometries map lines to lines.

Exercise 2.9 : Justify why the isometry $r_0 t_{(1,0)}$ is not the product of one or two reflections.

To reach to the theorem that any isometry is the product of one, two or three reflections, we need to observe that any isometry is determined by its effect on a triangle.

Lemma 2.10. *Any isometry f of $\mathbb{R}^2$ is determined by the images $f(A), f(B)$ and $f(C)$ of three points A, B, C not in a line.*

Corollary 2.11. *If L is the line of points equidistant from the points P and Q, then reflection in L exchanges P and Q.*

Theorem 2.12. *Any isometry of $\mathbb{R}^2$ is the product of one, two or three reflections.*

As a consequence of the theorem, we can observe that the isometries of $\mathbb{R}^2$ forms a group under composition, denoted by $\text{Iso}(\mathbb{R}^2)$. Further, consider the classes $\text{Iso}^+(\mathbb{R}^2)$ and $\text{Iso}^+(\mathbb{R}^2).r_0 =$\{product of odd numbers of reflections\}. By theorem **??**, part (**??**), we know that $\text{Iso}^+(\mathbb{R}^2)$ consists of rotations and translations. The fixed point set of a non trivial rotation is a single point and the fixed point set of a translation is empty, whereas the fixed point set of a reflection is a line. Hence, $\text{Iso}^+(\mathbb{R}^2).r_0$ is not $\text{Iso}^+(\mathbb{R}^2)$, implying that the products of even number of reflections forms a subgroup $\text{Iso}^+(\mathbb{R}^2)$ of index 2.
It is intuitively clear that the product of an even number of reflections preserves the sense of a clockwise oriented circle in $\mathbb{R}^2$, whereas the product of an odd number of reflections reverse it.

Exercise 2.13 : Show that the translations form a group but that the rotations do not.

By the above theorem, we know that every isometry is product of reflections, we distinguish the isometries into orientation preserving isometries(product of even number of reflections) and oreientation reversing isometries(product of odd number of reflections). Further, orientation preserving isometries are nothing but either a rotation or translation. Now, the question is what about orientation reversing isometries? That is, what about the product of odd number of reflections? We can see that it is either a reflection of product of three reflections. Thus, we need to obtain more clear description for an isometry which is product of three reflections. interestingly it becomes product of a reflection with a translation(which is product of two reflections) in the direction of the line of reflection, which we called as *glide reflection.* (See Example **??**). For example, the glide refnection when the axis is $x-$ axis becomes $f = t_{(\alpha,0)} s_0$ for some $\alpha \in \mathbb{R}$ or in complex form $f(z) = \alpha + \overline{z}$. Thus, we have the following theorem:

Theorem 2.14. *A product $s_L s_M s_N$ of reflections in lines L, M, N is a glide reflection.*

With this theorem, we now have the final remark about any isometry in $\mathbb{R}^2$ as follows:
Each isometry in $\mathbb{R}^2$ is either a rotation, translation or glide reflection.

Exercise 2.15 : Deduce from the classification that each euclidean isometry has exactly one of the following:

(1) A line of fixed points.
(2) A single fixed point.
(3) No fixed points, and a parallel family of invariant lines (an invariant line is a line mapped onto itself by the isometry).
(4) No fixed points, and a single invariant line.

3. Locally Euclidean Surfaces

In this section, the following question will be answered: which unbounded surfaces look locally like the euclidean plane $\mathbb{R}^2$? Although, $\mathbb{R}^2$ is intended to model "flat" surfaces in the real world, yet all physical flat surfaces are of finite spread and have boundaries. Such surfaces, if extended indefinitely, will they resemble $\mathbb{R}^2$, even if small parts of it matches with small parts of $\mathbb{R}^2$. This takes us to the idea of manifolds. In brief,, an n-dimensional manifold is a space S in which each point has a neighbourhood "like" an open ball in the euclidean space $\mathbb{R}^n$. At one extreme, they may be merely homeomorphic, in which case S is a *topological* manifold. On the other hand, we will be considering the other extreme, where the neighbourhoods are isometric. In this case, S is a *euclidean* manifold. Let us consider some examples:

3.1. **The Cylinder.** A cylinder can be made by joining the edges of a strip of paper, which is a part of a plane. Hence, cylinder is "locally like" the plane $\mathbb{R}^2$. We take a strip S of $\mathbb{R}^2$ bounded by parallel lines, say $x = 0$ and $x = 1$ and say that points on the cylinder C are the points of S, where points $(0, y)$ and $(1, y)$ are same points on C. (this "joins" the two edges of the strip). However, this construction is inelegant because we do not have same "situation" for all the points of C. Some points of C have two different corresponding points on the strip. We need to modify the construction of C so as to treat all points of the cylinder equally. For this we will be using all points of $\mathbb{R}^2$ instead of just a strip.
Along with each points (x, y) in the strip, we take all points $(x + n, y)$ to represent the same point of C. Intuitively speaking, we form C by "rolling up" the whole plane. (FIGURE **??**)
The process can be described precisely as a construction of the *quotient space* or *orbit space* $\mathbb{R}^2/\Gamma$, where Γ is the group of integer horizontal translations

of $\mathbb{R}^2$. In fact, $\Gamma = \langle t_{(1,0)} \rangle = \{t^n_{(1,0)} : n \in \mathbb{Z}\}$. A point of $C = \mathbb{R}^2/\Gamma$ is a set of the form $\{(x+n, y) : n \in \mathbb{Z}\} = \{t^n_{(1,0)}(x, y) : n \in \mathbb{Z}\}$.

[xscale =1.5, yscale = 0.7] [thick, ¡-¿](3, 0) –(0, 0) – (0,2); [thick, ¡-¿](-3,0) – (0, 0) – (0, -2); in -2,-1,0,1,2 [shift=(,0),color=black] (0pt,50pt) – (0pt,-50pt); in -2,-1,0,1,2 (,-0.3) node[left] ; in -1.8,-.8,0.2,1.2,2.2 (,1.6) node $*$; figure

For $P = (x, y) \in \mathbb{R}^2$, we denote this corresponding point on C by ΓP. To visualize the orbit space $\mathbb{R}^2/\Gamma$, we focus on *fundamental region*. It is a part of the plane which contain a representative of each Γ-orbit with at most one representative of each Γ-orbit in its interior.
The distance between two points ΓP and ΓQ from C is given by

$$d_C(\Gamma P, \Gamma Q) = \min\{d(P', Q') : P' \in \Gamma P, Q' \in \Gamma Q\}$$

where d is the euclidean metric on $\mathbb{R}^2$. Observe that each $P' \in \Gamma P$ has the same set of distances to the members of ΓQ. Thus, we can also write this as:

$$d_C(\Gamma P, \Gamma Q) = \min\{d(P, Q') : Q' \in \Gamma Q\}.$$

This expression shows that d_C is well defined because for each $P \in \mathbb{R}^2$, there is a nearest $Q \in \Gamma Q$ (possibly one of a pair that are equally near). We also note that if $d(P, Q) < 1/2$ we have $d_C(\Gamma P, \Gamma Q) = d(P, Q)$ since in this case, Q is nearest to P amongst all the members of ΓQ.
The *orbit map* is the map which sends $P \in \mathbb{R}^2$ to its orbit $\Gamma P \in C = \mathbb{R}/\Gamma$. We denote it by f. From the above observation, f maps every disc $D \subset \mathbb{R}^2$ of diameter $< 1/2$ isometrically into $\mathbb{R}^2/\Gamma$. Hence, f is a *local isometry*. Figure **??** gives an intuitive picture of the orbit map by representing an orbit (set of stars in $\mathbb{R}^2$) belonging to the abstract cylinder $\mathbb{R}^2/\Gamma$ by a point (single star) belonging to a concrete cylinder in space.

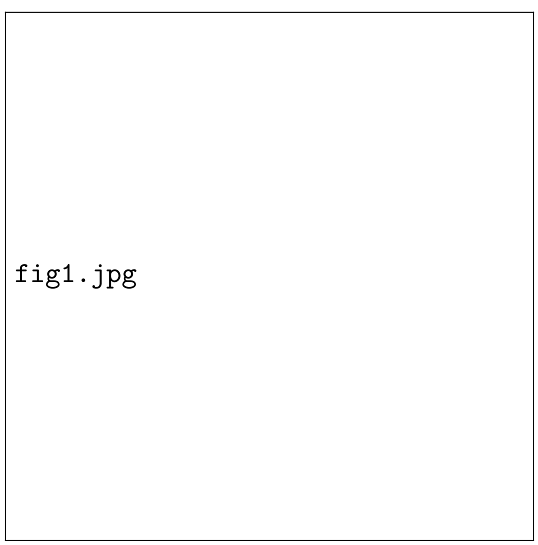

figure Γ sends all stars in $\mathbb{R}^2$ to a single star on the cylinder and hence Γ "wraps the plane around the cylinder". We say that f is a *covering* of C by $\mathbb{R}^2$. The definition of C as $\mathbb{R}^2/\Gamma$ has the advantage of giving the most direct definition of distance on C. The local isometry property of the orbit map f means that, within discs of diameter $< 1/2$, the geometry of the cylinder is the same as the geometry of the plane. However, interesting differences emerge when we try to extend geometric concepts to the whole cylinder. For example, it is natural to define a *line* on C to be the f - image of a line on $\mathbb{R}^2$. Such "lines" are "locally" the same as ordinary lines. That is, their intersection with any disc of diameter $< 1/2$ is a line segment. But, they can be globally quite different. There are three distinct types of lines on the cylinder illustrated in the below (FIGURE **??**):

fig3.jpg

figure

Exercise 3.1 : Which of the following properties of euclidean lines hold for lines on the cylinder?

(1) There is a line through any two points.
(2) There is a unique line through any two points.
(3) Two lines meet in at most one point.
(4) There are lines which do not meet.
(5) A line has infinite length.
(6) A line gives the shortest distance between two points.
(7) A line does not cross itself.

Exercise 3.2 : If $t_{(\alpha,\beta)}$ is a nontrivial translation of $\mathbb{R}^2$, and Γ is the group $\langle t_{(\alpha,\beta)} \rangle$ generated by $t_{(\alpha,\beta)}$, define $\mathbb{R}^2/\Gamma$ and show that it is the same as the cylinder above, up to a change of scale.

Exercise 3.3 : Show that a disc D of diameter $2/3$ is mapped one-to-one into C by the orbit map, but that $d_C(\Gamma P, \Gamma Q) \neq d(P, Q)$ for certain $P, Q \in D$.

An important advantage of the quotient construction is to discuss the following examples of euclidean surfaces, which *cannot* be represented properly in three dimensional space.

3.2. **The twisted cylinder.** The twisted cylinder C^* is constructed by joining opposite sides of a parallel-sided strip S, but with a twist. The resulting surface cannot lie in ordinary three-dimensional space without intersecting itself, though a fairly representative part of it can. This part is the Möbius band M, obtained by joining opposite sides of a rectangle R with a half twist(Figure **??**).

fig4.jpg

figure The twisted cylinder is obtained by prolonging the transverse segments of R. We formally define twisted cylinder as a quotient $\mathbb{R}^2/\Gamma$ for suitable Γ. Let us take strip S as the fundamental region of a group Γ generated by a glide reflection, which reflects in the x-axis and translates along the x-axis by 1 units. That is, Γ is a group generated by $t_{(1,0)}s_0$. Hence, $\Gamma = \langle t_{(1,0)}s_0 \rangle$.

[xscale =1.5, yscale = 0.7] [thick, ¡-¿](3, 0) –(0, 0) – (0,2); [thick, ¡-¿](-3,0) – (0, 0) – (0, -2); in -2,-1,0,1,2 [shift=(,0),color=black] (0pt,50pt) –

(0pt,-50pt); in -2,-1,0,1,2 (,-0.3) node[left] ; in -.8,1.2 (,-1.6) node $*$; in -1.8,0.2,2.2 (,1.6) node $*$; figure

All "stars" in figure **??** becomes a single star on the twisted cylinder. More precisely, Γ-orbit of any point (x, y) becomes $\{(x+n, (-1)^n y) : n \in \mathbb{Z}\}$. We define the points of C^* to be the Γ-orbits of points of $\mathbb{R}^2$. Further, as in case of a cylinder, we define the distance between any two points $\Gamma P, \Gamma Q$ of C^* as :

$$d_{C^*}(\Gamma P, \Gamma Q) = \min\{d(P, Q') : Q' \in \Gamma Q\}.$$

This makes the twisted cylinder to be locally euclidean. Infact, the discs of diameter less than 1/2 becomes isometric.

3.3. **The Torus and the Klein Bottle.** The torus is usually viewed as the doughnut-shaped surface obtained by rotating a circle in space. Such a surface can also be obtained by opposite sides of a rectangle. However, we can see that the distances on the rectangle are distorted by this construction. Since, we are interested in distance preserving constructions, we once again use the idea of quotient $\mathbb{R}^2/\Gamma$ to define the torus. This will atleast carry out the local geometry of $\mathbb{R}^2$. We consider the group generated by the translations $t_{(1,0)}$ and $t_{(0,1)}$. Thus, let $\Gamma = \{t^n_{(1,0)} t^m_{(0,1)} : m, n \in \mathbb{Z}\}$. The Klein bottle is related to the torus in much the same way as twisted cylinder is related to the cylinder. The usual construction is by joining opposite sides of a rectangle, with one pair of sides being joined with a twist. Figure ???? shows what happens to the rectangle whose sides have been labeled and directed so that the successive steps can be followed more easily. Like-labeled sides have to be joined, with their arrows pointing in the same direction. We define Klein bottle as a quotient $\mathbb{R}^2/\Gamma$ as follows:

The rectangle is the fundamental region for a group Γ generated by a glide reflection $g = t_{(1,0)} s_0$ in the horizontal direction and a translation $t_{(0,1)}$ in the vertical direction.

As with previous surfaces $\mathbb{R}^2/\Gamma$, we have an orbit map $f : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ defined by $f(P) = \Gamma P$ and a distance function on $\mathbb{R}^2/\Gamma$ which gives each orbit $P \in \mathbb{R}^2/\Gamma$ a neighbourhood ismoetric to a euclidean disc.

We have seen four different locally euclidean surfaces of the form $\mathbb{R}^2/\Gamma$, by choosing proper group Γ. We now show that there are no more euclidean surfaces $\mathbb{R}^2/\Gamma$ by showing that any other group Γ yields a quotient which is not a surface.

We call Γ *discontinuous* if no $P \in \mathbb{R}^2$ has a $\Gamma-$ orbit with a limit point. Further, we say Γ is *fixed point free* if $gP \neq P$ for each $P \in \mathbb{R}^2$ and each $g \neq 1$ in Γ.

Lemma 3.4. *If Γ is a group of isometries of $\mathbb{R}^2$, then Γ is discontinuous and fixed point free if and only if each $P \in \mathbb{R}^2$ has a neighborhood D_P in which each point belongs to a different Γ-orbit.*

Proof. Suppose Γ is discontinuous and fixed point free, and consider any $P \in \mathbb{R}^2$. Since Γ is discontinuous, there is a $\delta > 0$ such that all points in

the Γ-orbit of P are at a distance $\geq \delta$ from P. Then, since Γ is fixed point free, gP is at a distance $\geq \delta$ from P for each $g \neq 1$ in Γ. Thus, the whole neighborhood D_P of P with radius $\delta/3$ is shifted to a position disjoint from D_P by g, hence D_P cannot contain two points in same Γ-orbit.
Conversely, suppose each $P \in \mathbb{R}^2$ has a neighborhood D_P in which each point belongs to a different Γ-orbit. Then Γ must be discontinuous, otherwise some $P \in \mathbb{R}^2$ would have members of the same Γ-orbit in all its neighborhoods.
Also, Γ must be fixed point free. If not, consider a fixed point Q of some $g \neq 1$ in Γ. Since $g \neq 1$, g cannot be the identity on any neighborhood of Q (otherwise it would fix three points not in a line and hence would be the identity by the lemma **??**). Thus, g moves points R which are arbitrarily close to Q, and the gR are equally close to $gQ = Q$ because g is an isometry. In other words, any neighborhood D_Q of Q includes distinct points R, gR in the same Γ-orbit, contrary to hypothesis. □

Now, let us examine which of the isometries have no fixed points.

Isometry	**Fixed points**
Reflection	line of reflection
Rotation	point of rotation
translation	No fixed points
proper glide reflection	No fixed points

Thus, we know that only translations and proper glide reflections have no fixed points and hence Γ can include only these. Thus, we now have the following theorem:

Theorem 3.5. *A discontinuous, fixed point free group Γ of isometries of $\mathbb{R}^2$ is generated by one or two elements.*

Corollary 3.6. *$S = \mathbb{R}^2/\Gamma$ is a cylinder, twisted cylinder, torus or Klein bottle.*

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